

# Computing Check Digits – Fast

Adám Brudzewsky



# Overview

$$\{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \times 2$$



$$\{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) + . \times \omega\} \times 2$$

Speed-up by a factor of 500



# Overview

- One day in the APL Orchard (apl.chat)...
- A user presented an implementation
- “recommendations on how to improve it are very welcome”
- Challenge is on!





MINISTÉRIO DA FAZENDA  
Secretaria da Receita Federal

# CPF

**Cadastro de Pessoas Físicas**

**Número de Inscrição**

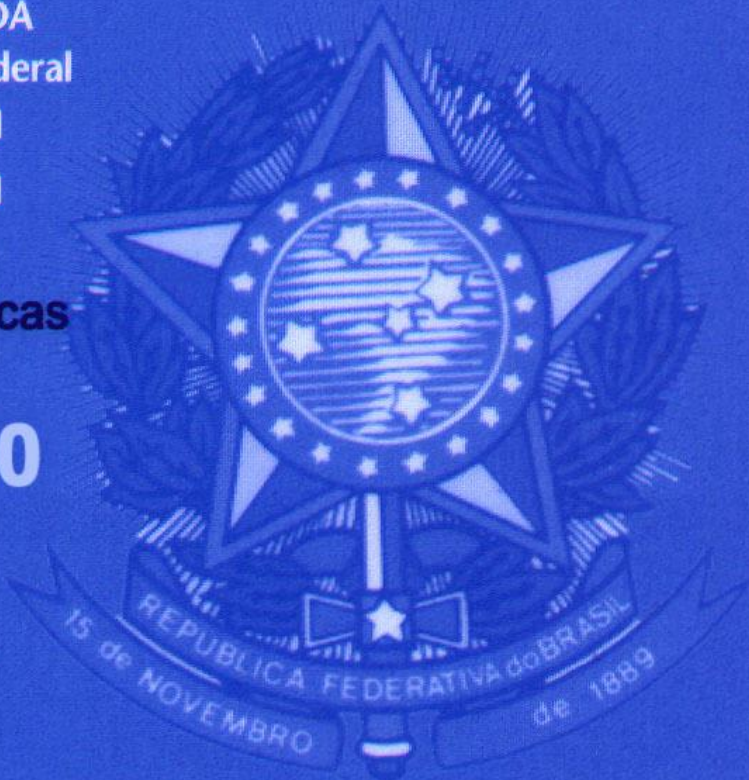
**000.000.000-00**

**Nome**

**NOME DA PESSOA**

**Nascimento**

**01/01/1990**





MINISTÉRIO DA FAZENDA  
Secretaria da Receita Federal

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**Número de Inscrição**

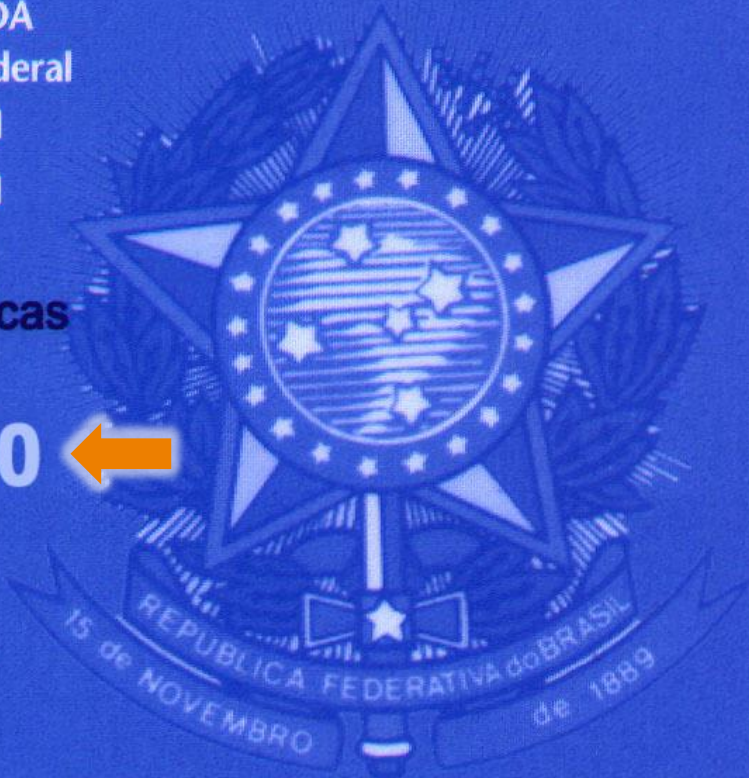
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**Nome**

**NOME DA PESSOA**

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**01/01/1990**



# Overview

**000.000.000-00**



# Overview

**314.159.265-90**



# Overview

**314 159 265 90**





# Overview

**314 159 265 9**



# Overview

**314 159 265**



# Algorithm Overview

1 Begin with nine digits

**3 1 4 1 5 9 2 6 5**



# Algorithm Overview

- 1 Begin with nine digits
- 2 Compute weights

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>
10	9	8	7	6	5	4	3	2



# Algorithm Overview

- 1 Begin with nine digits
- 2 Compute weights
- 3 Multiply digits with weights

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>
10	9	8	7	6	5	4	3	2
30	9	32	7	30	45	8	18	10



# Algorithm Overview

- 1 Begin with nine digits
- 2 Compute weights
- 3 Multiply digits with weights
- 4 Sum mod 11

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>
10	9	8	7	6	5	4	3	2
30	9	32	7	30	45	8	18	10
2								



# Algorithm Overview

- 1 Begin with nine digits
- 2 Compute weights
- 3 Multiply digits with weights
- 4 Sum mod 11
- 5 If 0 or 1, check digit is 0

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>
10	9	8	7	6	5	4	3	2
30	9	32	7	30	45	8	18	10
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—								



# Algorithm Overview

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10	9	8	7	6	5	4	3	2
30	9	32	7	30	45	8	18	10
2								
—								
9								





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10	9	8	7	6	5	4	3	2
30	9	32	7	30	45	8	18	10
2								
—								
9								
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>



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**3 1 4 1 5 9 2 6 5**

**10 9 8 7 6 5 4 3 2**

**30 9 32 7 30 45 8 18 10**

**2**

**—**

**3 1 4 1 5 9 2 6 5 9**



# Algorithm Overview

- 1 Begin with nine digits
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- 8 Repeat steps 2–7 with ten digits

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**3 1 4 1 5 9 2 6 5 9**

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11	10	9	8	7	6	5	4	3	2

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>



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11	10	9	8	7	6	5	4	3	2
33	10	36	8	35	54	10	24	15	18

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>



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11	10	9	8	7	6	5	4	3	2
33	10	36	8	35	54	10	24	15	18
									1

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>



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11	10	9	8	7	6	5	4	3	2
33	10	36	8	35	54	10	24	15	18
									1
									0
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>



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11	10	9	8	7	6	5	4	3	2
33	10	36	8	35	54	10	24	15	18
									1
									0
									—
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>





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11	10	9	8	7	6	5	4	3	2
33	10	36	8	35	54	10	24	15	18
									1

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	9	0
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	9	



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11	10	9	8	7	6	5	4	3	2
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<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>	<b>0</b>
<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>	<b>9</b>	



# Algorithm Implementation

<b>3</b>	<b>1</b>	<b>4</b>	<b>1</b>	<b>5</b>	<b>9</b>	<b>2</b>	<b>6</b>	<b>5</b>
10	9	8	7	6	5	4	3	2



# Algorithm Implementation

**3 1 4 1 5 9 2 6 5**  $\times 10$  9 8 7 6 5 4 3 2



# Algorithm Implementation

**+ / 3 1 4 1 5 9 2 6 5**  $\times 10$  9 8 7 6 5 4 3 2



# Algorithm Implementation

11 | + / 3 1 4 1 5 9 2 6 5 × 10 9 8 7 6 5 4 3 2



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$





# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$

$t \leq 1 : 0$


$11 - t$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$   
 $11 - t$

$0(11 - t)$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$

$t \leq 1 : 0$   
 $11 - t$

$(2 - t \leq 1) \Rightarrow 0(11 - t)$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$

$t \leq 1 : 0$   
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$(\underline{2 - t \leq 1}) \Rightarrow 0(11 - t)$



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$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$

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 $11 - t$

$(\underline{2 - t \leq 1}) \Rightarrow 0(11 - t)$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$

$t \leq 1 : 0$   
 $11 - t$

$(\underline{2 - t \leq 1}) \Rightarrow \underline{0} (11 - t)$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$   
 $11 - t$


$(2 - \underline{t \leq 1}) \triangleright 0 (11 - t)$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$   
 $11 - t$

$(2 - t \leq 1)$   $\Rightarrow 0(11 - t)$





# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$   
 $11 - t$

$(\underline{2 - t \leq 1}) \Rightarrow 0(\underline{11 - t})$

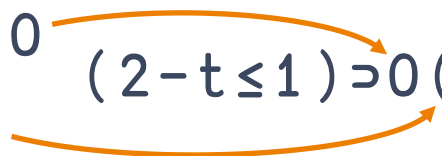


# Algorithm Implementation

$t \leftarrow 11 \mid + / \mathbf{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$

$t \leq 1 : 0$   
 $11 - t$

$(2 - t \leq 1) \Rightarrow 0(11 - t)$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \overbrace{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5}^{\omega} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$   
 $11 - t$

$(2 - t \leq 1) \Rightarrow 0(11 - t)$



# Algorithm Implementation

$$\begin{array}{cccccccccccccccc}
 & 3 & 1 & 4 & 1 & 5 & 9 & 2 & 6 & 5 \\
 & \omega & & \omega & & & & & & & \\
 t \leftarrow 11 \mid + / & \underbrace{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5}_{\omega} & \times & \underbrace{10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2}_{\omega}
 \end{array}$$

$$\begin{array}{l}
 t \leq 1 : 0 \\
 11 - t
 \end{array}
 \quad
 \begin{array}{l}
 \xrightarrow{\quad} (2 - t \leq 1) \Rightarrow 0(11 - t) \\
 \xrightarrow{\quad}
 \end{array}$$



# Algorithm Implementation

$$\begin{array}{l}
 t \leftarrow 11 \mid + / \overbrace{3 \ 1 \ 4 \ 1 \ 5 \ 9 \ 2 \ 6 \ 5}^{\omega} \times \overbrace{10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2}^{9 \neq \omega} \\
 t \leq 1 : 0 \quad (2 - t \leq 1) \Rightarrow 0(11 - t) \\
 11 - t
 \end{array}$$



# Algorithm Implementation

$$\begin{array}{c}
 \begin{array}{cccccccccccccccc}
 & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 & & & & & & & & & & \omega & \neq \omega & & & & & & & \\
 & & & & & & & & & & \underbrace{\hspace{10em}} & \underbrace{\hspace{10em}} & & & & & & & \\
 t \leftarrow 11 \mid + / & \mathbf{3} & \mathbf{1} & \mathbf{4} & \mathbf{1} & \mathbf{5} & \mathbf{9} & \mathbf{2} & \mathbf{6} & \mathbf{5} & \times & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2
 \end{array} \\
 t \leq 1 : 0 & \xrightarrow{\hspace{10em}} & (2 - t \leq 1) \Rightarrow 0(11 - t) \\
 11 - t & \xrightarrow{\hspace{10em}} &
 \end{array}$$



# Algorithm Implementation

$t \leftarrow 11 \mid + / \overbrace{3 \ 1 \ 4 \ 1 \ 5}^{\omega} \overbrace{9 \ 2 \ 6 \ 5}^{\phi_i \neq \omega} \times 10 \ 9 \ 8 \ 7 \ 6 \ 5 \ 4 \ 3 \ 2$   
 $t \leq 1 : 0$   
 $11 - t$   
 $(2 - t \leq 1) \Rightarrow 0(11 - t)$



# Algorithm Implementation

$$\begin{array}{cccccccccccccccc}
 & & & & & & & & & & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
 & & & & & & & & & & & & & & 1 + \phi \neq \omega & & & & \\
 & & & & & \omega & & & & & & & & & & & & & \\
 t \leftarrow 11 \mid + / & \mathbf{3} & \mathbf{1} & \mathbf{4} & \mathbf{1} & \mathbf{5} & \mathbf{9} & \mathbf{2} & \mathbf{6} & \mathbf{5} & \times & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\
 t \leq 1 : 0 & & & & & & & & & & & & & & & & & & & \\
 11 - t & & & & & & & & & & & & & & & & & & & 
 \end{array}$$

$(2 - t \leq 1) \Rightarrow 0(11 - t)$





# Algorithm Implementation

$$(2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega)$$



# Algorithm Implementation

$$\{\omega, (2-t \leq 1) \Rightarrow 0(11-t \leftarrow 11 \mid + / (1+\phi \neq \omega) \times \omega)\}$$



# Algorithm Implementation

$$\{\omega, (2-t \leq 1) \Rightarrow 0(11-t \leftarrow 11 \mid + / (1+\phi \neq \omega) \times \omega)\} \underline{\ast 2}$$



# Algorithm Implementation

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times \underline{2}$$



# Algorithm Implementation

$CPF \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

CPF 3 1 4 1 5 9 2 6 5



# Algorithm Implementation

$CPF \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \times 2$

CPF 3 1 4 1 5 9 2 6 5

3 1 4 1 5 9 2 6 5 9 0



# Algorithm Implementation

$CPF \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

CPF 3 1 4 1 5 9 2 6 5

3 1 4 1 5 9 2 6 5 9 0



# Algorithm Optimisation

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$





# Algorithm Optimisation

$$\text{CPF} \leftarrow \{ \omega, (2 - \underline{t \leq 1}) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$



# Algorithm Optimisation

$$\text{CPF} \leftarrow \{ \omega, (2 - \underbrace{t}_{\downarrow} \leq \underbrace{1}) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \wr \neq \omega) \times \omega) \} \ast 2$$



# Algorithm Optimisation

$$\text{CPF} \leftarrow \{ \omega, (2 - \underbrace{t \leq 1}) \rightarrow \underbrace{0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)} \} \times 2$$



# Algorithm Optimisation

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$

$$(2 - t \leq 1) \rightarrow 0 \quad 42$$



# Algorithm Optimisation

CPF  $\leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

t  $\leftarrow$  3     $\diamond$      $(2 - t \leq 1) \rightarrow 0$     42

t  $\leftarrow$  2     $\diamond$      $(2 - t \leq 1) \rightarrow 0$     42

t  $\leftarrow$  1     $\diamond$      $(2 - t \leq 1) \rightarrow 0$     42

t  $\leftarrow$  0     $\diamond$      $(2 - t \leq 1) \rightarrow 0$     42



# Algorithm Optimisation

$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$t \leftarrow 3 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42$

42

$t \leftarrow 2 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42$

42

$t \leftarrow 1 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42$

0

$t \leftarrow 0 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42$

0



# Algorithm Optimisation

$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi t \neq \omega) \times \omega) \} \times 2$

$t \leftarrow 3 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42 \quad \rightarrow \quad (t > 1) \times 42$

$42$   
 $t \leftarrow 2 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42 \quad \rightarrow \quad (t > 1) \times 42$

$42$   
 $t \leftarrow 1 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42 \quad \rightarrow \quad (t > 1) \times 42$

$0$   
 $t \leftarrow 0 \quad \diamond \quad (2 - t \leq 1) \rightarrow 0 \quad 42 \quad \rightarrow \quad (t > 1) \times 42$

$0$



# Algorithm Optimisation

CPF  $\leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi_i \neq \omega) \times \omega) \} \times 2$

t  $\leftarrow$  3    $\diamond$    (t > 1)  $\times$  42

42  
t  $\leftarrow$  2    $\diamond$    (t > 1)  $\times$  42

42  
t  $\leftarrow$  1    $\diamond$    (t > 1)  $\times$  42

0  
t  $\leftarrow$  0    $\diamond$    (t > 1)  $\times$  42

0





# Algorithm Optimisation

CPF  $\leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

t  $\leftarrow$  3     $\diamond$     (t > 1)  $\times$  42

42

t  $\leftarrow$  2     $\diamond$     (t > 1)  $\times$  42

42

t  $\leftarrow$  1     $\diamond$     (t > 1)  $\times$  42

0

t  $\leftarrow$  0     $\diamond$     (t > 1)  $\times$  42

0



# Algorithm Optimisation

$$\text{CPF} \leftarrow \left\{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \right\} \times 2$$

(t > 1) ×



# Algorithm Optimisation

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$

$$\text{CPF} \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \} \times 2$$



# Algorithm Optimisation

$CPF \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{ \omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \} \times 2$

$t \leftarrow 0$

$t \leftarrow 1$

$t \leftarrow 2$

$t \leftarrow :$

$t \leftarrow 9$

$t \leftarrow 10$



# Algorithm Optimisation

$CPF \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{ \omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \} \times 2$

11 ← 11 - t ← 0

10 ← 11 - t ← 1

9 ← 11 - t ← 2

⋮ ← 11 - t ← ⋮

2 ← 11 - t ← 9

1 ← 11 - t ← 10



# Algorithm Optimisation

$CPF \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0 (11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{ \omega, \quad \underline{(t > 1) \times} \quad 11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \quad \} \times 2$

0  $\leftarrow$  11  $\leftarrow$  11 - t  $\leftarrow$  0

0  $\leftarrow$  10  $\leftarrow$  11 - t  $\leftarrow$  1

9  $\leftarrow$  9  $\leftarrow$  11 - t  $\leftarrow$  2

$\vdots$   $\leftarrow$   $\vdots$   $\leftarrow$  11 - t  $\leftarrow$   $\vdots$

2  $\leftarrow$  2  $\leftarrow$  11 - t  $\leftarrow$  9

1  $\leftarrow$  1  $\leftarrow$  11 - t  $\leftarrow$  10



# Algorithm Optimisation

$CPF \leftarrow \{\omega, (2-t \leq 1) \rightarrow 0(11-t \leftarrow 11 \mid + / (1+\phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, \quad \underline{(t > 1)} \times 11-t \leftarrow 11 \mid + / (1+\phi \neq \omega) \times \omega \} \times 2$

0  $\leftarrow$  11  $\leftarrow$  11-t  $\leftarrow$  0

0  $\leftarrow$  10  $\leftarrow$  11-t  $\leftarrow$  1

9  $\leftarrow$  9  $\leftarrow$  11-t  $\leftarrow$  2

$\vdots$   $\leftarrow$   $\vdots$   $\leftarrow$  11-t  $\leftarrow$   $\vdots$

2  $\leftarrow$  2  $\leftarrow$  11-t  $\leftarrow$  9

1  $\leftarrow$  1  $\leftarrow$  11-t  $\leftarrow$  10



# Algorithm Optimisation

$CPF \leftarrow \{\omega, (2-t \leq 1) \Rightarrow 0(11-t \leftarrow 11 \mid + / (1+\phi \neq \omega) \times \omega)\} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11-t \leftarrow 11 \mid + / (1+\phi \neq \omega) \times \omega\} \times 2$

0  $\leftarrow$  11

0  $\leftarrow$  10





# Algorithm Optimisation

0 ← 11

0 ← 10

$CPF \leftarrow \{\omega, (2-t \leq 1) \rightarrow 0(11-t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11-t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \} \times 2$



# Algorithm Optimisation

0 ← 11

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \ddot{*} 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \ddot{*} 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11



0 ← 1 1  
0 ← 1 0

1 2 3 4 5 6 7 8 9 10 11  
11- 11 | 11  
10 9 8 7 6 5 4 3 2 1 11



# Algorithm Optimisation

0 ← 11

0 ← 10

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$$

$$\text{CPF} \leftarrow \{ \omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$$

ι 11

1 2 3 4 5 6 7 8 9 10 11

11 - 11 | ι 11

10 9 8 7 6 5 4 3 2 1 11

11 | 11 - 11 | ι 11

10 9 8 7 6 5 4 3 2 1 0



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \ddot{*} 2$

$CPF \leftarrow \{ \omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \ddot{*} 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

11 - 11 |  $\iota$  11

10 9 8 7 6 5 4 3 2 1 11

11 | 11 - 11 |  $\iota$  11

10 9 8 7 6 5 4 3 2 1 0



0 ← 1 1 ✓  
0 ← 1 0

```

1  2  3  4  5  6  7  8  9 10 11
                                11 | 11-   11 | 11
10 9  8  7  6  5  4  3  2  1  0

```



0 ← 1 1 ✓  
0 ← 1 0

$$\begin{array}{cccccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 & & & & & & & & & 11 & | & - & 11 & | & 11 \\
 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0
 \end{array}$$


0 ← 1 1 ✓  
0 ← 1 0

```

1  2  3  4  5  6  7  8  9 10 11
                                11 | -      11
10 9  8  7  6  5  4  3  2  1  0

```



0 ← 1 1 ✓  
0 ← 1 0

1	2	3	4	5	6	7	8	9	10	11	
										11	
											-
10	9	8	7	6	5	4	3	2	1	0	



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

11 | -

$+ / (1 + \phi \iota \neq \omega) \times \omega$

10 9 8 7 6 5 4 3 2 1 0



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \} \times 2$   
 $i 11$

1 2 3 4 5 6 7 8 9 10 11

11 | -

10 9 8 7 6 5 4 3 2 1 0

$+ / (1 + \phi \neq \omega) \times \omega$



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

11 | -

10 9 8 7 6 5 4 3 2 1 0

+ / (1 +  $\phi \iota \neq \omega$ )  $\times \omega$



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2-t \leq 1) \Rightarrow 0(11-t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11-t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

11 | -

10 9 8 7 6 5 4 3 2 1 0

$+ / (1 + \phi \iota \neq \omega) \times \omega$



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

11 | -

10 9 8 7 6 5 4 3 2 1 0

+ / (1 +  $\phi \iota \neq \omega$ )  $\times \omega$



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega \} \times 2$   
 $i 11$

1 2 3 4 5 6 7 8 9 10 11

11 | -

+ / (1 +  $\phi \neq \omega$ )  $\times \omega$

10 9 8 7 6 5 4 3 2 1 0



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2-t \leq 1) \rightarrow 0(11-t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11-t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

11 |

10 9 8 7 6 5 4 3 2 1 0

$+ / (\underline{-1 - \phi \iota \neq \omega}) \times \omega$





# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \ddot{*} 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \ddot{*} 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

$11 \mid + / (-1 - \phi \iota \neq \omega) \times \omega$

10 9 8 7 6 5 4 3 2 1 0



0 ← 1 1 ✓  
0 ← 1 0

1 2 3 4 5 6 7 8 9 10 11

10 9 8 7 6 5 4 3 2 1 0

$$11 \mid + / (-1 - \phi_1 \neq \omega) \times \omega$$


# Algorithm Optimisation

0 ← 11 ✓

0 ← 10

$CPF \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$

$CPF \leftarrow \{\omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$   
 $\iota 11$

1 2 3 4 5 6 7 8 9 10 11

$(t \neq 10) \times$

$t \leftarrow 11 \mid + / (-1 - \phi \iota \neq \omega) \times \omega$

0 9 8 7 6 5 4 3 2 1 0



# Algorithm Optimisation

0 ← 11 ✓

0 ← 10 ✓

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega) \} \times 2$$

$$\text{CPF} \leftarrow \{ \omega, (t > 1) \times 11 - t \leftarrow 11 \mid + / (1 + \phi \iota \neq \omega) \times \omega \} \times 2$$

$$\iota 11$$

1 2 3 4 5 6 7 8 9 10 11

$(t \neq 10) \times$

$t \leftarrow 11 \mid + / (-1 - \phi \iota \neq \omega) \times \omega$

0 9 8 7 6 5 4 3 2 1 0



# Algorithm Optimisation

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$

$$\text{CPF} \leftarrow \{ \omega, \quad \quad \quad \} \times 2$$

$$(t \neq 10) \times \quad \quad t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega$$


# Algorithm Optimisation

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \times 2$$

$$\text{CPF} \leftarrow \{\omega, (t \neq 10) \times \quad t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \times 2$$


# Algorithm Optimisation

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} * 2$$
$$\text{CPF} \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} * 2$$


# Algorithm Optimisation

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \wr \neq \omega) \times \omega)\} \times 2$$
$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \wr \neq \omega) \times \omega\} \times 2$$




# Algorithm Comparison

$CPF \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \times 2$

$CPF_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \times 2$

$cpfs \leftarrow 10 \mid ? 1E6 \quad 9p10 \quad \text{a million CPFs}$



# Algorithm Comparison

$CPF \leftarrow \{\omega, (2-t \leq 1) \rightarrow 0(11-t \leftarrow 11 | + / (1+\phi \neq \omega) \times \omega)\} \times 2$

$CPF_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 | + / (-1-\phi \neq \omega) \times \omega\} \times 2$

$cpfs \leftarrow 10 | ? 1E6 \ 9p10$       A a million CPFs

`]RunTime -c (CPF°1)cpfs (CPF_a°1)cpfs`



# Algorithm Comparison

$CPF \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 | + / (1 + \phi \neq \omega) \times \omega)\} \times 2$

$CPF_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 | + / (-1 - \phi \neq \omega) \times \omega\} \times 2$

$cpfs \leftarrow 10 | ? 1E6 \quad 9p10 \quad \text{A a million CPFs}$

`]RunTime -c (CPF◊1)cpfs (CPFa◊1)cpfs`

$(CPF \circ 1)cpfs \rightarrow 2.6E0 \quad | \quad 0\% \quad \square\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square$

$(CPF_a \circ 1)cpfs \rightarrow 1.9E0 \quad | \quad -27\% \quad \square\square\square\square\square\square\square\square\square\square$



# Algorithm Extension

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \times 2$$
$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \times 2$$


# Algorithm Extension

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \ast 2$$
$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \ast 2$$
$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \ast 2$$


# Algorithm Extension

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \ast 2$$
$$\text{CPF}_a \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega \} \ast 2$$
$$\text{CPF}_b \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid \underline{+ / (-1 - \phi \neq \omega) \times \omega} \} \ast 2$$


# Algorithm Extension

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \ast 2$$
$$\text{CPF}_a \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega \} \ast 2$$
$$\text{CPF}_b \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) + \underline{\cdot} \times \omega \} \ast 2$$


# Algorithm Extension

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \ast 2$$
$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \ast 2$$
$$\text{CPF}_b \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) + \underline{\cdot} \times \omega\} \ast 2$$




# Algorithm Extension

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega)\} \times 2$$
$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega\} \times 2$$
$$\text{CPF}_b \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) \times \omega\} \times 2$$


# Algorithm Extension

$$\text{CPF} \leftarrow \{\omega, (2 - t \leq 1) \Rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi_i \neq \omega) \times \omega)\} \times 2$$

$$\text{CPF}_a \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi_i \neq \omega) \times \omega\} \times 2$$

$$\text{CPF}_b \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_i \vdash / \rho \omega) + . \times \sim \omega\} \times 2$$


# Algorithm Comparison

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$

$$\text{CPF}_a \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi \neq \omega) \times \omega \} \times 2$$

$$\text{CPF}_b \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) + . \times \omega \} \times 2$$

]RunTime -c (CPF $\circ$ 1)cpfs (CPF<sub>a</sub> $\circ$ 1)cpfs CPF<sub>b</sub> $\vdash$ cpfs



# Algorithm Comparison

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi_i \neq \omega) \times \omega) \} \ddot{*} 2$$
$$\text{CPF a} \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid + / (-1 - \phi) \neq \omega \} \div 2$$
$$\text{CPFb} \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{\tau} / \rho \omega) + . \times \ddot{\omega} \} \ddot{2}$$

```
]RunTime -c (CPF $\ddot{0}$ 1)cpfs (CPF $\alpha\ddot{0}$ 1)cpfs CPFb $\vdash$ cpfs
```

(CPF÷1)cpfs → 2.8E0 | 0%

(CPFa<sup>0</sup>1)cpfs → 1.9E0 | -29% 

CPFb $\vdash$ cpfs  $\rightarrow 5.2E^{-2}$  | -98%



# Algorithm Optimisation

$$\text{CPF}b \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{t-1} / \rho\omega) + . \times \omega\} \times 2$$



# Algorithm Optimisation

$CPFb \leftarrow \{\omega, (t \neq 10) \times (-1 - \phi_{11} / \rho\omega) + . \times \omega\} \times 2$



# Algorithm

CPF b ←

9	ρ	-10	-9	-8	-7	-6	-5	-4	-3	-2
16 9	ρ	3	1	4	1	5	9	2	6	5
		2	7	1	8	2	8	1	8	3
		1	6	1	8	0	3	3	9	9
		6	0	2	2	1	4	0	7	6
		2	9	9	7	9	2	4	5	8
		6	6	2	6	0	7	0	1	5
		1	6	0	2	1	7	6	6	3
		1	3	7	0	3	5	9	9	9
		1	4	1	4	2	1	3	5	6
		1	1	9	8	1	4	0	2	4
		1	0	5	9	4	6	3	0	9
		1	2	0	2	0	5	6	9	0
		1	3	2	4	7	1	7	9	6
		1	4	6	5	5	7	1	2	3
		1	7	3	2	0	5	0	8	1
		2	4	1	4	2	1	3	5	6

$\rho(\omega) + \rho(\omega) \times \omega \times 2$



# Algorithm Optimisation

9	16	p	3	2	1	6	2	6	1	1	1	1	1	1	1	1	1	2	9	p	-10
			1	7	6	0	9	6	6	3	4	1	0	2	3	4	7	4			-9
			4	1	1	2	9	2	0	7	1	9	5	0	2	6	3	1			-8
			1	8	8	2	7	6	2	0	4	8	9	2	4	5	2	4			-7
			5	2	0	1	9	0	1	3	2	1	4	0	7	5	0	2			-6
			9	8	3	4	2	7	7	5	1	4	6	5	1	7	5	1			-5
			2	1	3	0	4	0	6	9	3	0	3	6	7	1	0	3			-4
			6	8	9	7	5	1	6	9	5	2	0	9	9	2	8	5			-3
			5	3	9	6	8	5	3	9	6	4	9	0	6	3	1	6			-2

2





# Algorithm Optimisation

$CPFb \leftarrow \{\omega, (t \neq 10) \times (-1 - \phi_{11} / \rho\omega) + . \times \omega\} \times 2$



# Algorithm Optimisation

$$\text{CPFb} \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho\omega) + . \times \ddot{\omega}\} \times 2$$



# Algorithm Optimisation

$\text{CPFb} \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho\omega) + . \times \ddot{\omega}\} \times 2$   
 $\text{CPFb} \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho\omega) + . \times \ddot{\omega}\} \times 2$



# Algorithm Optimisation

$$\text{CPF}_b \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho \omega) + . \times \omega \} * 2$$

$$\text{CPF}_c \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} \neq \omega) + . \times \omega \} * 2$$



# Algorithm Optimisation

$$\begin{aligned}\text{CPF}_b &\leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho\omega) + . \times \omega\} \times 2 \\ \text{CPF}_c &\leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} \neq \omega) + . \times \omega\} \times 2 \\ \text{cpfst} &\leftarrow \text{cpf}_s\end{aligned}$$


# Algorithm Comparison

$CPFb \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho\omega) + . \times \omega\} \times 2$

$CPFc \leftarrow \{\omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} \neq \omega) + . \times \omega\} \times 2$

$cpfst \leftarrow \ominus cpfs$

]RunTime -c  $CPFb \vdash cpfs$   $CPFc \vdash cpfst$



# Algorithm Comparison

$$\text{CPFb} \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho \omega) + . \times \ddot{\omega} \} \ddot{2}$$
$$\text{CPF}_C \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_t \neq \omega) + . \times \omega \} \div 2$$

$\text{cpfst} \leftarrow \emptyset \text{cpfs}$

```
]RunTime -c CPFb←cpfs CPFc←cpfst
```

CPFb+cpfs → 4.3E-2 | 0% 

```
* CPFc-cpfst → 2.3E-2 | -46% ██████████
```



# Algorithm Comparison

$$\text{CPFb} \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_{11} / \rho \omega) + . \times \ddot{\omega} \} \ddot{2}$$
$$\text{CPF}_C \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_t \neq \omega) + . \times \omega \} \div 2$$

cnf ← hcpfs

5 s for  
all of Brazil

## CPFb-cpfs

CPFc+cpfst

CPFb+cpfs 4.3E-2 | 0% 

\* CPFc-cpfst  $\rightarrow 2.3E^{-2}$  | -46% 





# Algorithm Compilation

$$\text{CPF } c \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_i \neq \omega) + . \times \omega \} \times 2$$

*5 s for  
all of Brazil*

$\text{CPF } c \leftarrow \text{cpf } st \rightarrow 2.3\text{E}^{-2} \mid 0\% \square$



# Algorithm Compilation

$$\text{CPF}_C \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_t \neq \omega) + . \times \omega \} \div 2$$



5 s for  
all of Brazil

CPFc+cpfst  $\rightarrow$  2.3E-2 | 0% 



# Algorithm Compilation

$$\text{CPF}_C \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_t \neq \omega) + . \times \omega \} \div 2$$



5 s for  
all of Brazil



## Co-dfns

CPF<sub>c</sub> ← c<sub>pfst</sub> → 2.3E-2 | 0%



# Algorithm Compilation

$$\text{CPF}_c \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_i \neq \omega) + . \times \omega \} * 2$$


*5 s for  
all of Brazil*



Co-dfns

$\text{CPF}_c \vdash \text{cpf st}$	$\rightarrow 2.3\text{E}^{-2}$		0%	□□□□□□□□□□□□□□□□□□
$\text{CPF}_d \vdash \text{cpf st}$	$\rightarrow 4.8\text{E}^{-3}$		-79%	□□□



# Algorithm Compilation

$$\text{CPF}_c \leftarrow \{ \omega; (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi_i \neq \omega) + . \times \omega \} \times 2$$


*5 s for  
all of Brazil*

*1 s for  
all of Brazil*



Co-dfns

$\text{CPF}_c \vdash \text{cpf}_{st} \rightarrow 2.3\text{E}^{-2}$		0%	□□□□□□□□□□□□□□□□□□
$\text{CPF}_d \vdash \text{cpf}_{st} \rightarrow 4.8\text{E}^{-3}$		-79%	□□□



# Algorithm Comparison

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$

$$\text{CPF}_c \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) + . \times \omega \} \times 2$$


# Algorithm Comparison

$$\text{CPF} \leftarrow \{ \omega, (2 - t \leq 1) \rightarrow 0(11 - t \leftarrow 11 \mid + / (1 + \phi \neq \omega) \times \omega) \} \times 2$$


$$\text{CPF}_c \leftarrow \{ \omega, (t \neq 10) \times t \leftarrow 11 \mid (-1 - \phi \neq \omega) + . \times \omega \} \times 2$$

	a million CPFs	per CPF	all of Brazil
Original	> 2.5 s	> 2.5 μs	> 8 min
	↓	↓	↓
Optimised	< 25 ms	< 25 ns	< 5 s
	↓	↓	↓
Compiled	< 5 ms	< 5 ns	< 1 s



# Lessons Learned





# Lessons Learned: Ask Yourself...



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- Can you use mathematics instead of selection?



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- ✧ Can you use mathematics instead of selection?
- ✧ Can you simplify the formulas?



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- ✧ Can you process multiple inputs at once?



# Lessons Learned: Ask Yourself...

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- ✧ If so, can you use inner products?



# Lessons Learned: Ask Yourself...

- ✧ Can you use mathematics instead of selection?
- ✧ Can you simplify the formulas?
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- ✧ If so, can you use inner products?
- ✧ How is the data is stored?



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- ✧ Can the steps be done on vectors?



# Lessons Learned: Ask Yourself...

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- ✧ If so, can you use inner products?
- ✧ How is the data is stored?
- ✧ Can the steps be done on vectors?
- ✧ Does it run faster if you transpose your data?





# Lessons Learned: Ask Yourself...

- Can you use mathematics instead of selection?
- Can you simplify the formulas?
- Can you process multiple inputs at once?
- If so, can you use inner products?
- How is the data is stored?
- Can the steps be done on vectors?
- Does it run faster if you transpose your data?
- Will compilation speed it up?



# Upcoming webinars

Jan	27	<b>BAA</b>	open session	<a href="http://britishaplassociation.org">britishaplassociation.org</a>
Feb	10	<b>BAA</b>	APL logo	<a href="http://see.apl.wiki/APL_logo">see apl.wiki/APL_logo</a>
Feb	17	<b>Dyalog</b>	(topic TBA)	<a href="http://dyalog.tv">dyalog.tv</a>
Feb	24	<b>BAA</b>	open session	
Mar	10	<b>BAA</b>	open session	
Mar	17	<b>Dyalog</b>	(topic TBA)	
Mar	24	<b>BAA</b>	open session	



# Thanks for Watching!

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