Abstract

Computation on large Boolean arrays is becoming more prevalent, due to applications such as cryptography, data compression, and image analysis and synthesis. The advent of bit-oriented vector extensions for microprocessors and of GPUs presents opportunities for significant performance improvements in such Boolean-dominated applications. Since APL is one of the few computer languages that supports dense (one bit per element, eight bits per byte), multi-dimensional Boolean arrays as first-class objects, it has naturally attracted research into optimizations for improved performance of Boolean array operations. This paper presents some of the Single Instruction, Multiple Data (SIMD) Boolean-related optimizations that have appeared in APL implementations, and suggests ways in which those optimizations might be exploited using contemporary hardware.

Why Does APL have One-bit Booleans?

- Blame Larry Breed: while designing APL\360,
- Breed decided to store Booleans densely, eight bits/byte
- Booleans were stored in row-major order, as are other arrays
- This eased indexing, structural and selection verbs, etc.
- Single-bit indexing was more expensive than word indexing…
- But it opened the door to SIMD Boolean array optimizations
- Those optimizations are the subject of this talk
- Speedups were usually 8X or 32X, but sometimes even more
- A half century later, Breed’s decision remains brilliant
- These optimizations are still important and relevant
- GPU and SIMD vector facilities can exploit them

A BIT of Introduction

- The bit: the fundamental unit of digital computing
- Yet, few computer languages treat bits as basic data types
- Fewer support multi-dimensional bit arrays (8 bits/byte)
- Fewer yet provide array operations on Boolean arrays
- Boolean arrays appear in image analysis, cryptography, data compression…
- The burden of bit twiddling is left to the programmer
- APL, however, simply treats Booleans as the integers 0 and 1
- Boolean arrays are grist to APL’s data-parallel, expressive mill!

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Scalar Verbs

- Breed optimized many rank-0 (scalar) Boolean verbs e.g.
- Boolean verbs: ∧, ∨, ¬, ※, ∨ ...
- Relational verbs: <, ≤, ≥, >, ≠
- SIMD application, a word at a time (32 bits on S/360)
- One or more of us optimized scalar extension, e.g.
- 1∧B would produce B,
- without doing any element-wise computations

Strength Reduction: a Classic Compiler Optimization

- Strength reduction: replace one operation by a cheaper one
- E.g., replace multiply by a power of two with a shift
- In APL, Boolean B1×B2 becomes B1∧B2
- In APL, Boolean B1\B2 becomes B1∧B2
- In APL, Boolean B1+B2 becomes B1≥B2
- Performance boosts: simpler verbs, SIMD operation, no conditionals, stay in Boolean domain
- Performance boosts: In a compiler, opportunity for other optimizations

Structural and Selection Verbs I

- catenate, laminate, rotate, reverse, rank, from,
- merge, take, drop...
- These verbs, e.g., 1 0 1, 0 1 1 0 have to handle array indices that are not byte-aligned
- We would like these to run SIMD, word-at-a-time, on Booleans
- We introduced rbremove: generalized stride-1 (ravel order) copier verb
- snk[snr+k]+src[sri+k]
- Does not corrupt out-of-bounds array elements
- Operates in SIMD mode(s) whenever possible
- Supports all type conversions

Structural and Selection Verbs II

- Operation on non-trailing array axes:
- SIMD copy entire subarrays at once, e.g.
  1 0 2 3 4 5 2 4
- rbremove will copy 12 adjacent array elements at once
Structural and Selection Verbs III

- 2 3 4 12 4
  0 1 2 3
  4 5 6 7
  8 9 10 11

- 12 13 14 15
- 16 17 18 19
- 20 21 22 23

Reverse and Rotate on Booleans

- Bernecky, 1979: fast algorithms for Φω & αΦω
- Last-axis Boolean Φω did a byte at a time, w/table lookup...
  RevTab[uint8 ω]
- Then byte-aligned the resulting vector, SIMD, a word at a time
- All non-last-axis operations copied entire cells at once, using rbemove

Reverse and Rotate Performance on Booleans

Reshape

- Reshape allows element reuse, e.g.:
  8P1 0 0
  1 0 0 1 0 0 1 0
- Breed optimized Boolean reshape this way:
- Copy the argument to the result
- Catenate the partial result to itself, doubling its length, until its tail is byte-aligned.
- Do an overlapped move, or "smear" of the result to its tail
### Transpose I

- An unlikely candidate for SIMD, it would seem...
- $a \otimes b$ with unchanged trailing axes
- $T+2 \times 2 \times 3 \times 24$
  
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### Transpose II

- Transpose with unchanged trailing axes
- SIMD copy six elements at once (rbemove)
- $T \times 0 \times 2 \times 3 \times 24$
  
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### Boolean Transpose III

- *Hacker's Delight*: fast 8x8 Boolean matrix transpose
- Kemel: 16 logical & shift operations on 64-bit ravel
- Uses *perfect shuffle* (PDEP) on any power of two shape
- Dyalog APL (Foad): 10X speedup on large Boolean array transpose
- Kemel generalizes to any power of two, e.g., 16x16, 32x32

### Search Verbs

- Bernecky, 1971: fast `indexof` and `set membership`
- All data types except reals with `\ct\neq 0`
- `(a\otimes 1)[w]`
- `(a\otimes a\vee)[w]`
- Booleans: Vector search for first byte of interest
- Then, table lookup to get bit offset
- Speedup: lots - linear time vs. quadratic time
- Created `indexof` kernel utility for interpreter use
Reduce

- Roger D. Moore, 1971: fast \(+/\omega\) Boolean vector
- Initial use was compress and expand setup:
  \(+/\alpha\) was taking longer than compress/expand
- S/360 translate vector op: Boolean bytes into population counts, SIMD 124 bytes per segment
- SIMD integer sum of 4-byte words gave 4-element partial sum
- Shift-and-add gave final result
- Segment size limited to prevent inter-byte carries
- Larry Breed haiku:
  \(+/\neq/4\) resh PopcountTab\[uint8 \omega\]
- Algorithm used briefly for \(\lor/\omega\) and \(\land/\omega\)

Reduce and Scan

- E.E. McDonnell, 1974: elegant APL models of Boolean scan and reduction for relational
- Result was catenation of prefix, infix, & suffix expressions
- Used Bernekey’s fast indexof
- Result: linear-time, word-at-a-time, SIMD Boolean scan & reduce

Scan

- John Heckman, 1970 or 1971: user-defined APL scan verbs
- Now widely used in GPUs
- Recursive doubling:
  \(r\leftarrow\text{nescanall }y; s; biw\)
  \(\alpha\) Not-equal scan
  \(r\leftarrow y\)
  \(\text{biw} \leftarrow \text{2*1}\)\(\pi y\)
  :For \(s\) :In 2*\text{biw} \(\alpha\) Heckman
  \(r \leftarrow r \neq (s-r) \leftarrow (s-r) \leftarrow r\)
  :EndFor
- SIMD, word-at-a-time algorithm for Boolean \(\neq/\omega\) and \(=/\omega\) along last axis
- Bernekey’s simple C Heckman implementation is about 3X faster than Dyalog APL 15.0 (vector only)
- So far, no X86 vectorization; perhaps we can do even better

STAR Inner Product

- IPSA, 1973: Boolean array inner products were painfully slow
- Control Data (CDC) wanted APL for their new STAR-100 vector supercomputer
- Group from Toronto I.P. Sharp Associates hired to work on the interpreter
- Memory-to-memory vector instructions needed stride-1 access for good performance
- Bernekey: heard about STAR APL stride-1 inner-product algorithm;
  redesigned Boolean inner product to use STAR algorithm
Classic Inner Product Algorithm

\[ Z \text{ipclassic} Y;RX;CX;CY;I;J;K \]
\[ RX \cdot (\rho X)[0] \]
\[ CX \cdot (\rho X)[1] \]
\[ CY \cdot (\rho Y)[1] \]
\[ Z \cdot (RX, CY) \cdot 0.5 \]
:For \( I \) :In \ RX
   :For \( J \) :In \ CY
     Z[I;J] = 0
   :EndFor
:EndFor
:EndFor

STAR Inner Product Algorithm

\[ Z \text{ipstar} Y;RX;CX;CY;I;J;Xel \]
\[ RX \cdot (\rho X)[0] \]
\[ CX \cdot (\rho X)[1] \]
\[ CY \cdot (\rho Y)[1] \]
\[ Z \cdot (RX, CY) \cdot 0 \]
:For \( I \) :In \ RX
   :For \( J \) :In \ CY
     Xel = X[I;J]
     Z[I;J] = Z[I;J] + Xel \cdot Y[J;]
   :EndFor
:EndFor

STAR Inner Product II

- Inner product loops reordered; key benefits, for \( f . g . \omega \)
- Each \( a \) element, Xel, fetched only once
- Type conversion of Xel no longer time-critical
- Xel analysis amortized over entire row: \( Y[J;] \)
- Scalar-vector application of \( g \)
  \( \text{tmp} = Xel \cdot Y[J;] \)
- Vector-vector \( f \)-reduce into result row \( Z[I;] \)
  \( Z[I;] = Z[I;] \cdot f \text{ tmp} \)

SIMD Boolean STAR Inner Product Basics

- Scalar-vector \( Xel \cdot g \cdot Y[J;] \) is word-at-a-time Boolean SIMD
- Vector-vector \( Z[I;] \cdot Z[I;] \cdot Xel \cdot Y[J;] \) is word-at-a-time Boolean SIMD
- We are already a lot faster
- The STAR APL model does \( + . \times 90X \) faster than the Classic model, on 200\times200 real matrices
- Unfortunately, the APL primitive is still 30X faster than the APL model
Boolean STAR Inner Product Optimizations

- Consider $\text{tmp} \cdot \text{Xel} \land \text{RO}$ in $\alpha \cdot \omega$
- If $\text{Xel}$ is 0, then $\text{tmp}$ is all zeros: no $g$ computation
- If $\text{Xel}$ is 1, then $\text{tmp}$ is just $\text{RO}$: no $g$ computation
- $f$ is $\vee$, so its identity element is 0
- Hence, if $\text{Xel}$ is 0, we can skip the $g$-reduction
- Similarly, if $\text{Xel}$ is 1, we can do the $g$-reduction using $\text{RO}$
- This gives us poor man's sparse arrays, which works on other data types, too
- Final result: Boolean inner products on SHARP APL/PC ran much faster than APL2 on huge mainframe

Control Flow Becomes Data Flow

- Booleans as arithmetic: replace control flow by data flow
- Conditionals can often be removed, e.g.:
- Give those with salary, $S$, less than Tiny a raise of $R \\ S + R > S < \text{Tiny}$
- Knuth calls this capability Iverson's convention for characteristic functions
- See also the verb $\text{mq}s$, for finding quoted text
- See also the Bernecky-Scholz PLDI2014 Arrays Workshop paper: Abstract Expressionism

Boolean Sort

- Sort ascending for Booleans:
  $\text{SortAscending} \cdot (\lnot \phi \circ \circ \oplus / \omega) \rho 1$
- Boolean sort can use Moore's SIMD $+/\text{Boolean}$ in its first phase of execution
- Second phase can be performed in SIMD, e.g., by a single SAC data-parallel with-loop.

Boolean Grade

- SIMD Boolean upgrade:
  $\text{ug} \cdot ((\lnot \omega) \circ \circ \omega \circ \circ \rho 1)$
- Not stunningly SIMD, though.
Boolean Matrix Operations

- Shard: For byte-oriented algorithms, a possibly empty sub-byte fragment of a matrix row, extending from the start of the row to next byte, or from the last byte in the row to the end of the row.
- Handling shards is a nuisance; it destroys algorithmic beauty
- Handling shards is also slower than beautiful algorithms
- Consider:
  1 1 0 0, 0 0 0 0, 1 0 1 0, 1 1 1 0
- The vector 1 0 1 is a shard, because its elements start at a byte boundary, but end in mid-byte.
- The vector 0 1 1 1 0 is a shard, because it starts in mid-byte, and ends on a byte boundary.
- A similar definition holds for word-oriented algorithms

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